Direct Serendipity Finite Elements on Cuboidal Hexahedra

Chuning Wang

Department of Mathematics

The University of Texas at Austin

Todd Arbogast, Mathematics, Oden Institute, UT-Austin

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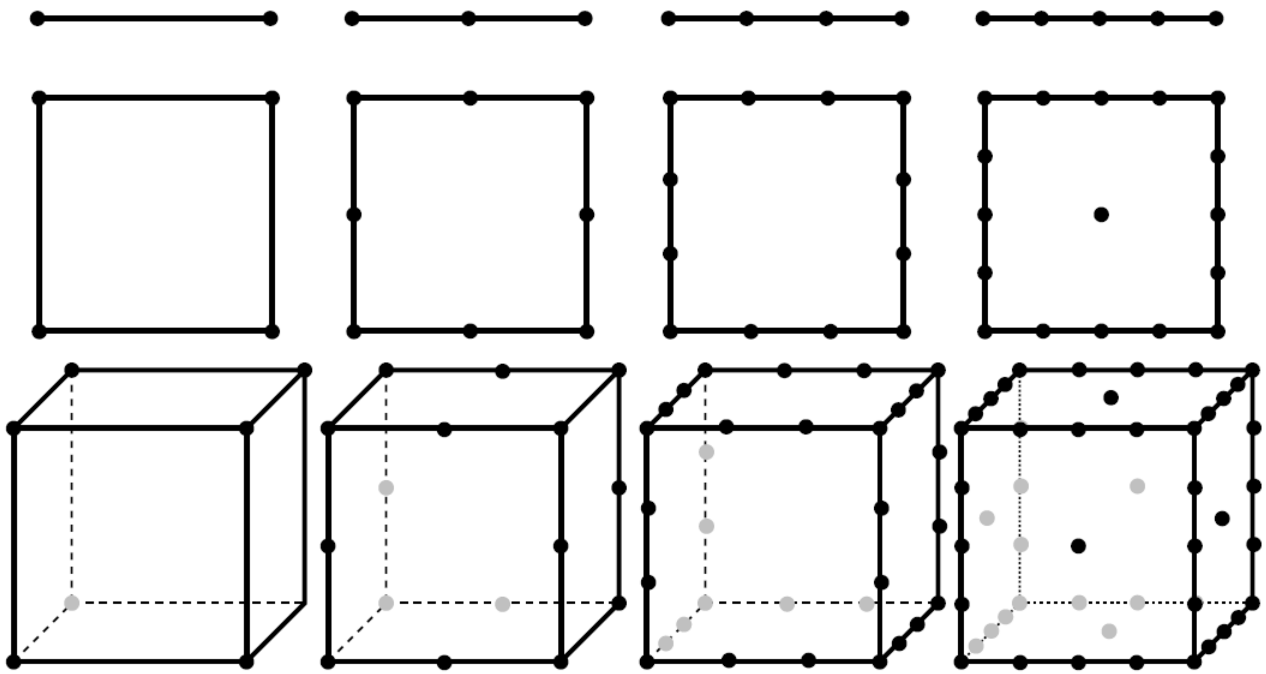
1. Review of Previous Results

Serendipity Space. The serendipity space $\mathcal{S}_r(I^n)$ is the space of all polynomials in n variables with superlinear degree at most r.

Degrees of freedom (DoFs).

$$u \mapsto \int_f uq, \quad q \in \mathbb{P}_{r-2d}(f)$$

Geometric Decomposition.



Properties.

- H^1-conforming
- Approximate to $\mathcal{O}(h^{r+1})$ with minimal number of DoFs

Problem. Lose accuracy when mapped to a quadrilateral or a general cuboidal hexahedron

Goals. Define direct finite element spaces that

- Include polynomials $\mathbb{P}_r$ *directly* in the space (for approximation)
- Use minimal number of DoFs

Previous Work.

- Construct the direct serendipity and mixed finite elements on quadrilaterals (Arbogast, Tao, & Wang 2022)
- Generalize to convex polygons (Arbogast & Wang 2022)

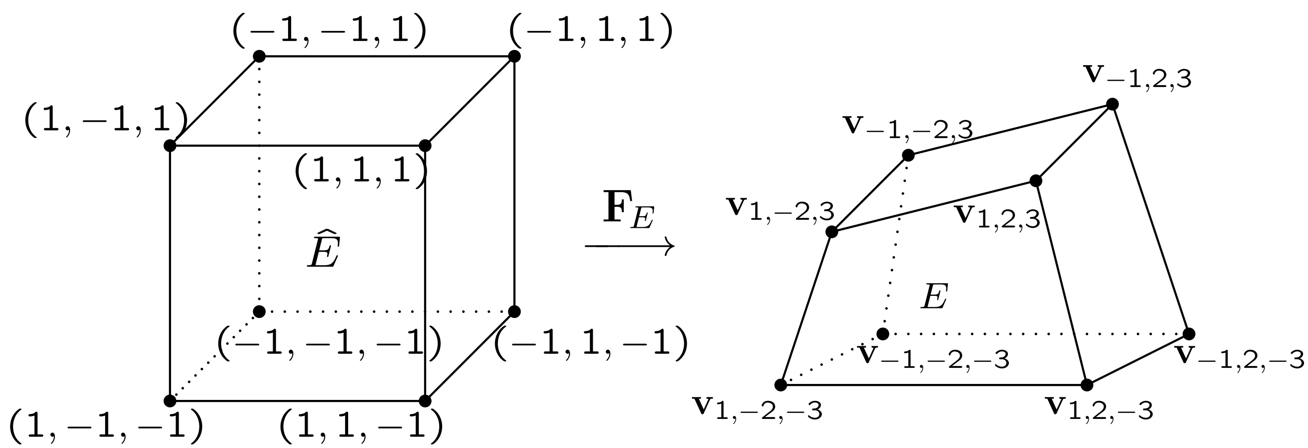
2. Direct Serendipity Finite Elements on Cuboidal Hexahedra

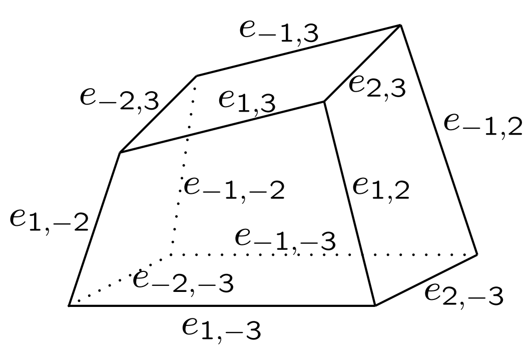
$$\mathcal{DS}_r(E) = \mathbb{P}_r(E) \oplus \mathbb{S}_r(E)$$

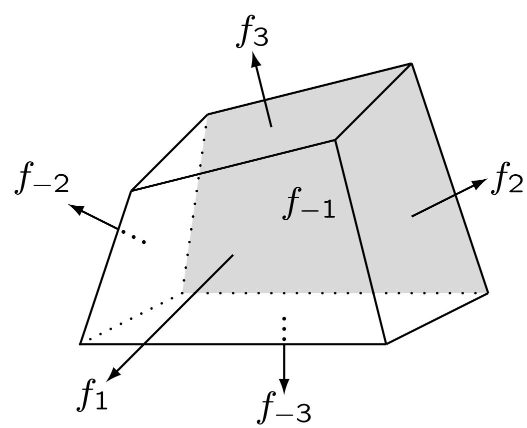
Polynomials plus supplements

Indexing

$$\hat{f}_{-1} = \hat{E} \cap \{\hat{x} = -1\} \xrightarrow{\mathbf{F}_E} f_{-1}, \quad \hat{f}_{1} = \hat{E} \cap \{\hat{x} = 1\} \xrightarrow{\mathbf{F}_E} f_{1},$$
$$\hat{f}_{-2} = \hat{E} \cap \{\hat{y} = -1\} \xrightarrow{\mathbf{F}_E} f_{-2}, \quad \hat{f}_{2} = \hat{E} \cap \{\hat{y} = 1\} \xrightarrow{\mathbf{F}_E} f_{2},$$
$$\hat{f}_{-3} = \hat{E} \cap \{\hat{z} = -1\} \xrightarrow{\mathbf{F}_E} f_{-3}, \quad \hat{f}_{3} = \hat{E} \cap \{\hat{z} = 1\} \xrightarrow{\mathbf{F}_E} f_{3}.$$







$$e_{i,j} = f_i \cap f_j, \quad e_{i,k} = f_i \cap f_k, \quad e_{j,k} = f_j \cap f_k, \quad \mathbf{v}_{i,j,k} = f_i \cap f_j \cap f_k \ .$$

Degrees of Freedom. Evaluation $\phi(\mathbf{v})$ at all the vertices $\mathbf{v}$, and for all the edges e and all the faces f,

$$\int_e \phi\, q, \quad \forall q \in \mathbb{P}_{r-2}(e), \qquad \int_f \phi\, q, \quad \forall q \in \mathbb{P}_{r-4}(f), \qquad \int_E \phi\, q, \quad \forall q \in \mathbb{P}_{r-6}(E).$$

DoFs Counting ($r < 3$).

$$D_r = \begin{cases} 8 = \dim \mathbb{P}_1(E) + 4, & \text{if } r = 1 \\ 20 = \dim \mathbb{P}_2(E) + 10, & \text{if } r = 2 \end{cases}$$

DoFs Counting ($r \geq 3$).

Dimension	Object	Number	DoFs/Object	Total DoFs
0	vertex	8	1	8
1	edge	12	$\dim \mathbb{P}_{r-2}(\mathbb{R})$	$12(r-1)$
2	face	6	$\dim \mathbb{P}_{r-4}(\mathbb{R}^2)$	$3(r-2)(r-3)$, if $r \geq 2$
3	interior	1	$\dim \mathbb{P}_{r-6}(\mathbb{R}^3)$	$\frac{1}{6}(r-3)(r-4)(r-5)$, if $r \geq 3$

We must add 3(r+1) supplements to $\mathbb{P}_r$

Linear Functions.

1. λ_n, for $n = \pm 1, \pm 2, \pm 3$:

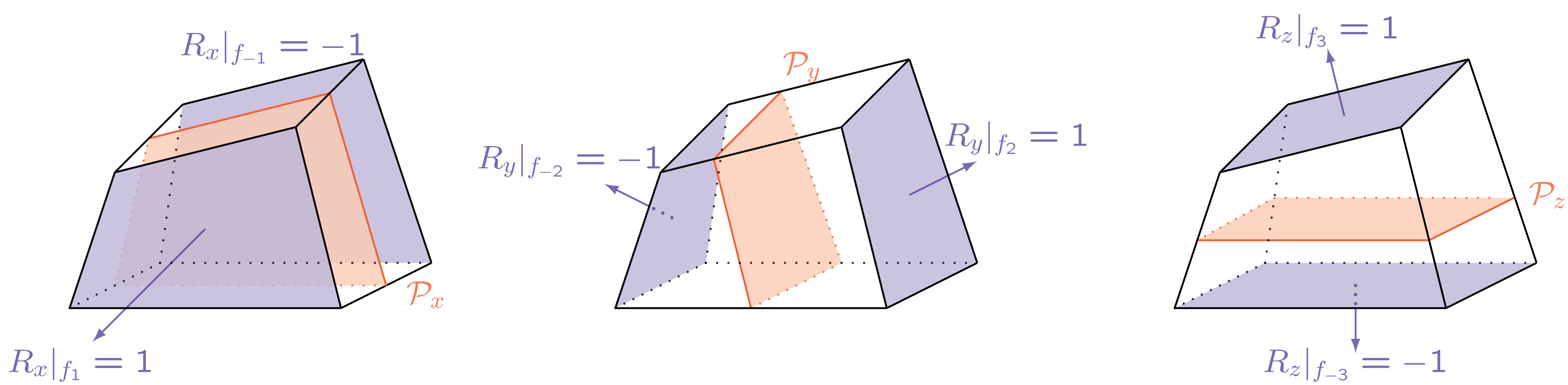
$$\lambda_n(\mathbf{x}) = -(\mathbf{x} - \mathbf{x}_{f_n}) \cdot \nu_n$$

   where $\mathbf{x}_{f_n} \in f_n$, and ν_n is the unit outer normal to face f_n.

2. For λ_x, λ_y, and λ_z, denote the zero plane of λ_* as $\mathcal{P}_*$:
   - $\mathcal{P}_x$ intersects $e_{\pm 2, \pm 3}$, $\mathcal{P}_y$ intersects $e_{\pm 1, \pm 3}$, $\mathcal{P}_z$ intersects $e_{\pm 1, \pm 2}$
   - $\mathcal{P}_x$, $\mathcal{P}_y$, and $\mathcal{P}_z$ do not coincide

Special maps.

$$R_x = \begin{cases} -1, & \text{on } f_{-1}, \\ 1, & \text{on } f_1, \end{cases} \quad R_y = \begin{cases} -1, & \text{on } f_{-2}, \\ 1, & \text{on } f_2, \end{cases} \quad R_z = \begin{cases} -1, & \text{on } f_{-3}, \\ 1, & \text{on } f_3. \end{cases}$$



$3(r-3)$ face supplements. For $s = 0, 1, \ldots, r-4$,

$$(1-\hat{y}^2)(1-\hat{z}^2)\hat{y}^s\hat{z}^{r-4-s}\hat{x} \quad \Rightarrow \quad \phi^f_{x,s} = \lambda_{-2}\lambda_2\lambda_{-3}\lambda_3\lambda_y^s\lambda_z^{r-4-s}R_x$$
$$(1-\hat{x}^2)(1-\hat{z}^2)\hat{x}^s\hat{z}^{r-4-s}\hat{y} \quad \Rightarrow \quad \phi^f_{y,s} = \lambda_{-1}\lambda_1\lambda_{-3}\lambda_3\lambda_x^s\lambda_z^{r-4-s}R_y$$
$$(1-\hat{x}^2)(1-\hat{y}^2)\hat{x}^s\hat{y}^{r-4-s}\hat{z} \quad \Rightarrow \quad \phi^f_{z,s} = \lambda_{-1}\lambda_1\lambda_{-2}\lambda_2\lambda_x^s\lambda_y^{r-4-s}R_z$$

12 edge supplements. If $i = \pm 1$ and $j = \pm 2$, the two edge basis functions $\phi^e_{i,j;s}$ for $s = r-3, r-2$ are defined as

$$\phi^e_{i,j;s}(\hat{x}, \hat{y}, \hat{z}) = \frac{1}{4}\hat{z}^s(1-\hat{z}^2)\,(1+\mathrm{sign}(i)\hat{x})\,(1+\mathrm{sign}(j)\hat{y})$$

The 4 functions with $s = r-2$ involve 3 supplements

$$(1-\hat{z}^2)\hat{z}^{r-2}\hat{x} \quad \Rightarrow \quad \phi^e_{z,1} = \lambda_{-3}\lambda_3\lambda_z^{r-2}R_x$$
$$(1-\hat{z}^2)\hat{z}^{r-2}\hat{y} \quad \Rightarrow \quad \phi^e_{z,2} = \lambda_{-3}\lambda_3\lambda_z^{r-2}R_y$$
$$(1-\hat{z}^2)\hat{z}^{r-2}\hat{x}\hat{y} \quad \Rightarrow \quad \phi^e_{z,3} = \lambda_{-3}\lambda_3\lambda_z^{r-2}R_xR_y$$

The definition of $\phi^e_{x,1}, \phi^e_{x,2}, \phi^e_{x,3}, \phi^e_{y,1}, \phi^e_{y,2}, \phi^e_{y,3}$ follows by symmetry.

However, the last 3 supplements $(1-\hat{x}^2)\hat{x}^{r-3}\hat{y}\hat{z}$, $(1-\hat{y}^2)\hat{y}^{r-3}\hat{x}\hat{z}$, and $(1-\hat{x}^2)\hat{x}^{r-3}\hat{y}\hat{z}$ related to $s = r-3$ do not generalize naturally.

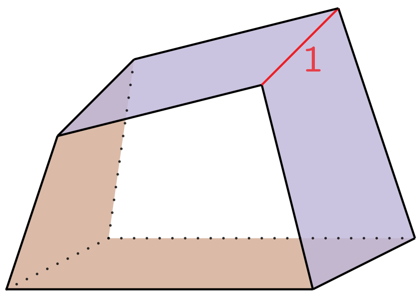
Generalization of the Supplements — 2

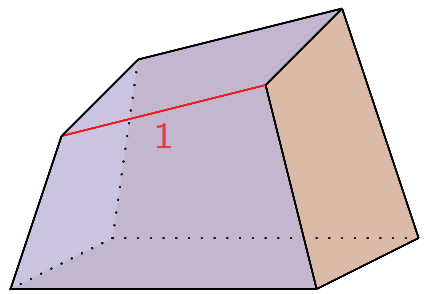
Idea. The restriction of the supplements to each face f should fall into the $\mathcal{DS}_r^{(2)}(f)$ defined on quadrilateral (Arbogast, Tao, & Wang 2022)

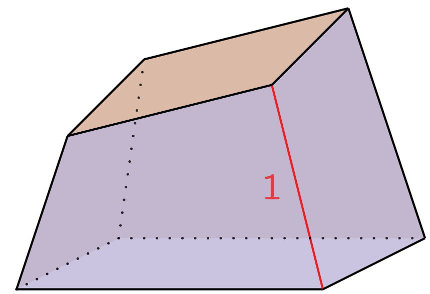
Supplemental functions.

$$\phi^e_{x,4} = \lambda_{-1}\lambda_1\lambda_x^{r-3}\psi_x, \qquad \phi^e_{y,4} = \lambda_{-2}\lambda_2\lambda_y^{r-3}\psi_y, \qquad \phi^e_{z,4} = \lambda_{-3}\lambda_3\lambda_z^{r-3}\psi_z$$

Requirements of ψ_x, ψ_y, and ψ_z on faces.







$$\psi_x|_{f_{-2}\cup f_{-3}} = 0 \qquad \psi_y|_{f_{-1}\cup f_{-3}} = 0 \qquad \psi_z|_{f_{-1}\cup f_{-2}} = 0$$

$$\psi_x|_{e_{2,3}} = 1 \qquad \psi_y|_{e_{1,3}} = 1 \qquad \psi_z|_{e_{1,2}} = 1$$

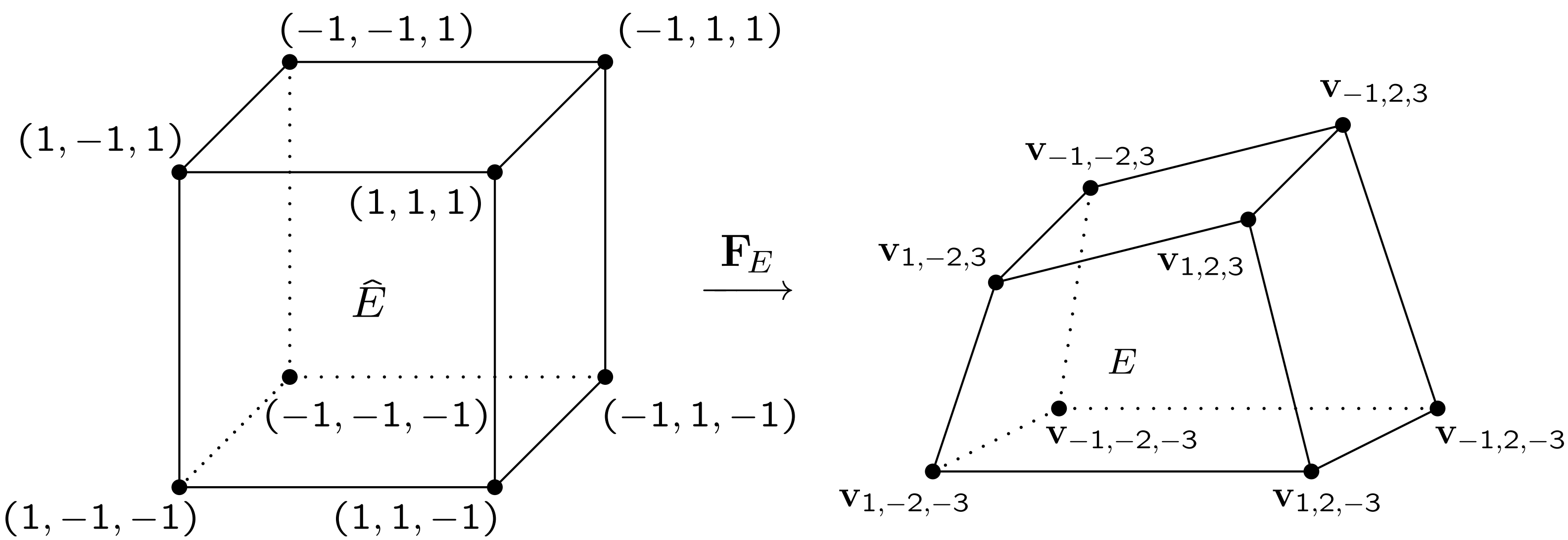
$$\psi_x|_{f_2} \in \mathbb{P}_1 \oplus \{\lambda_x R_z\} \qquad \psi_y|_{f_1} \in \mathbb{P}_1 \oplus \{\lambda_y R_z\} \qquad \psi_z|_{f_1} \in \mathbb{P}_1 \oplus \{\lambda_z R_y\}$$

$$\psi_x|_{f_3} \in \mathbb{P}_1 \oplus \{\lambda_x R_y\} \qquad \psi_y|_{f_3} \in \mathbb{P}_1 \oplus \{\lambda_y R_x\} \qquad \psi_z|_{f_2} \in \mathbb{P}_1 \oplus \{\lambda_z R_x\}$$

Pullback map.

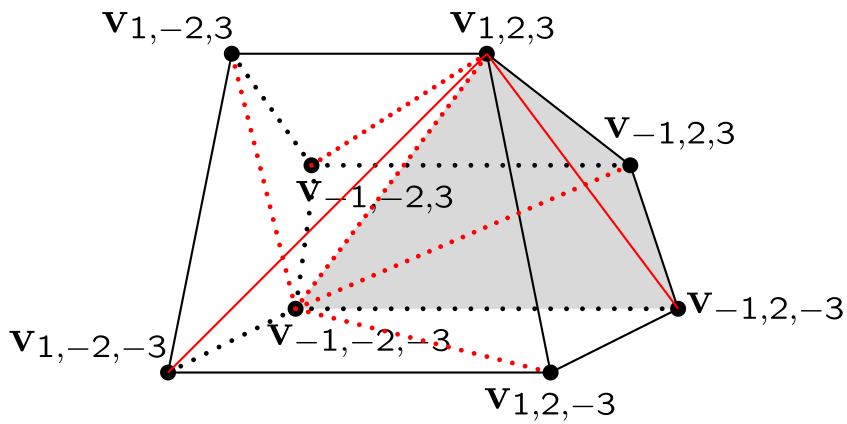
$$\psi_{x,2} = \psi_x|_{f_2}$$
$$\psi_{x,3} = \psi_x|_{f_3}$$

$$\psi_x(\mathbf{x}) = \widehat{\psi}_x(\widehat{\mathbf{x}}) = \widehat{\psi}_x(\widehat{x}, \widehat{y}, \widehat{z}) = \psi_{x,2}(\mathbf{F}_E(\widehat{x}, 1, \widehat{z}))\, \psi_{x,3}(\mathbf{F}_E(\widehat{x}, \widehat{y}, 1))$$

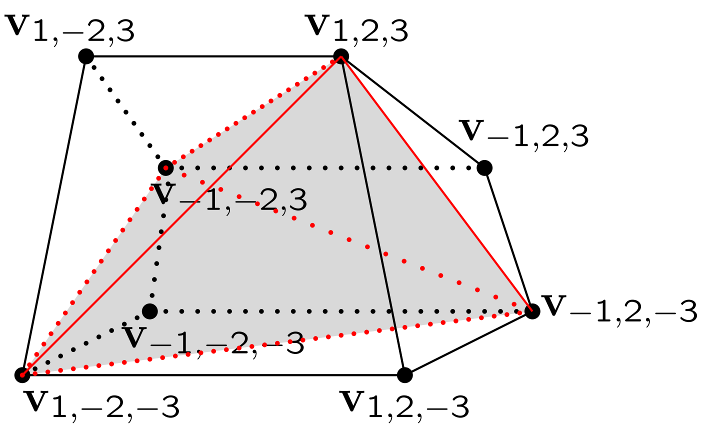


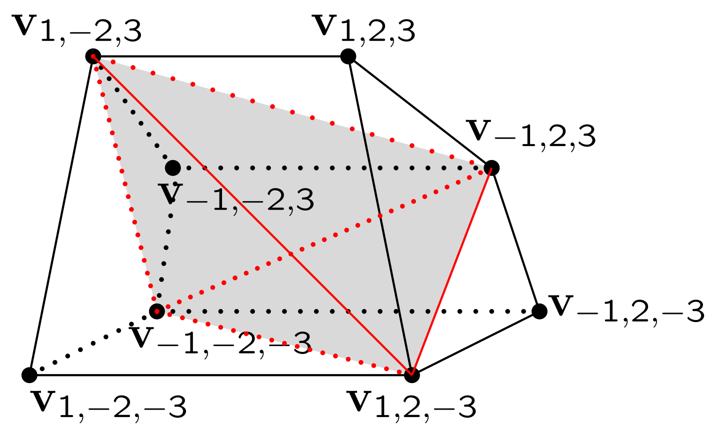
Piecewise polynomials.

1. Marching tetrahedra:



2. The diamond cubic based partition:





Vertex basis functions.

$$\phi^{\mathbf{v}}_{i,j,k} = \lambda_{-i}\lambda_{-j}\lambda_{-k} \in \mathbb{P}_r,$$

Cell basis functions.

$$\lambda_{-1}\lambda_1\lambda_{-2}\lambda_2\lambda_{-3}\lambda_3\,\mathbb{P}_{r-6} \in \mathbb{P}_r$$

Face basis functions. Face basis functions corresponding to $\mathbb{P}_s(f)$

- lie in $\mathbb{P}_r$ for $s \leq r-5$
- are linear combinations of $p \in \mathbb{P}_r$ and supplements for $s = r-4$

Edge basis functions. Edge basis functions

- lie in $\mathbb{P}_r$ for $s \leq r-4$
- are linear combinations of $p \in \mathbb{P}_r$ and supplements for $s = r-3, r-2$

Remarks

Lower dimensional spaces. For $r = 1, 2$, $\mathcal{DS}_r(E) \subset \mathcal{DS}_3(E)$.

Unisolvence. Unisolvence lies in the construction of shape functions.

Conformity. For any two adjacent elements sharing a common face f, we ask

- The zero planes of λ_x, λ_y, or λ_z that intersect f coincide on f;
- R_x, R_y, and R_z agree on f:
  - Marching tetrahedra: agree on the common faces naturally;
  - Diamond cubic: the two patterns must be used alternately by adjacent elements.

Remark. To form a global H^1-conforming basis, either solve a small linear system to match DoFs or use nodal DoFs instead.

3. Summary and Conclusions

1. Conforming finite elements are important in many areas.
   - Solving PDEs in certain applications.
   - General interpolation and approximation of functions.
   - Visualization.
2. Direct serendipity finite elements developed for cuboidal hexahedra.
   - H^1-conforming and fully constructive. Keys to the construction:
     - Form higher order spaces $r \geq 3$ to $r = 1, 2$.
     - Cell and vertex shape functions are straightforward.
     - Face and edge shape functions require supplements.
     - $\mathcal{DS}_r(E)|_f$ coincides $\mathcal{DS}_r^{(2)}(f)$.
   - Minimal DoFs and approximate optimally on shape regular meshes.
   - No accuracy loss due to reference element mapping.
3. Future work
   - Applications.
   - De Rham complex and direct mixed finite elements for cuboidal hexahedra.
   - Extension to more general 3D polytopes.

References

1. T. Arbogast and Chuning Wang: Direct serendipity and mixed finite elements on convex polygons, *Numer. Algorithms*, 2022.
2. T. Arbogast, Zhen Tao, and Chuning Wang: Direct serendipity and mixed finite elements on convex quadrilaterals. *Numer. Math.*, 2022.